1 Introduction

Explicit temporal state and input-output equations of networks and systems are essential in systems and control theory to study their behaviour and structural properties, such as their fixed points (steady-state solutions), stability, limit cycles, feedback design etc.. The field of System Science has covered many different types of systems from continuous time to discrete time, from linear to non-linear, from time invariant to time varying, discrete event driven systems to stochastic systems. System Science has contributed to many engineering concepts in chemical, nuclear, manufacturing, social systems, economics etc., and biological systems in molecular level; gene regulatory networks[16], [17], and even judicial processes are no exceptions. Many of these systems have been modeled as Boolean networks [31] [2] [28] [18] [14]. A rich literature has been created dealing with Boolean equations and network models in particular, and variety of systems have been modeled using Boolean networks in general.

Specifically introduced by negation operation, $\bar{\ }$, Boolean networks are highly non-linear finite dynamical systems. Starting from any initial state, their trajectories will either enter a periodic orbit with multiple points, or end in a fixed point in their state space. A number of literature review studies have been performed presenting the challenges manifested in such non-linear, logical dynamical systems as Boolean networks. For the present study, it has been very beneficial to revisit the previous work created by George Boole in 1850's [3], contributed by Shannon in 1940's [29], by numerous approaches published in late 1900's [19], early 2000's [1] [21], and lately in 2010's [30] [27] [12] [24]. The studies on many interesting system theoretic topics on Boolean networks still continue today [33] [22] [11] [4] [13] [30].

A matrix representation approach with semi-tensor-product, in an effort to utilize linear system theory concepts to study Boolean networks, has recently generated a lot of attention [6] [5] [7] [20]. Stability concepts were introduced into Boolean networks in publications as recent as those in 2017-2018 utilizing Graph Theory [13] [4]. Altough a comprehensive review of the field can be found in [10], an increasing amount of work on Boolean networks has since accumulated as noted above.

Differently from what has been done in the literature, in this paper, we interpret an already existing model of Boolean networks in greater detail, and identify their structure matrices which define their properties and temporal behaviour. We report different forms of matrix representations of Boolean networks with emphasis on structure matrices, and provide their cross relations. In order to accomplish these targets, Section 2 of the paper is devoted to some results from the existing literature, and we derive some extensions to those results in relation to Boolean variables and to the solutions of Boolean equations.

In section 3, we present and detail the model for Boolean networks and identify some structure matrices inherently in them. Utilizing the results of Section 2, we explore various properties of these structure matrices. Depending on the state variable chosen, we state various forms that result in input-state-output equations from the model, involving state, input and output, explicitly. We demonstrate that the reported forms of state-input-output equations can be manipulated to give more insight into steady state, stability, and feedback concepts in Boolean networks.

In general, temporal behaviour of a Boolean network, using Graph Theory, or Systems Theory point of view, has been described by a mixed state variable format, namely,

$$x(k+1) = R\, w(k)$$

where x represents the state, R is a matrix of 1's and 0's, and w contains the state x and the input u implicitly, in conjunctive canonical form [29]. Such evolution equations of Boolean