where matrix $Z_i$ is comprised of the columns of $R$, $r_{i+(j-1)*2^m}, j = 1, \ldots, 2^n$, for $i = 1, \ldots, 2^m$. Therefore, Form-2 reveals a property that every Boolean network has: If the input $u$ is kept constant at all times, the dynamics of the network is determined by a set of columns of $R$. As the input changes, the set of columns governing the network behaviour changes as different set of columns of $R$ take effect in accordance with the input. The network behaves one at a time per one of the $2^m$ number of $2^n \times 2^n$ $Z_i$ matrices where $\overline{\bar{M}\bar{Z}_i + MZ_i}$ matrix is $2^n \times 2^n$ stochastic Boolean matrix, $i = 1, \ldots, 2^m$, for which eigenvalue analysis can be done to identify the dynamics involved in the network.

The above forms of Boolean networks can then be summarized in $x$, $u$, and $w_x$, $w_u$, conjunctive canonical representation of $x$ and $u$, as follows

$$\begin{aligned} x(k+1) &= F_1(w(k)) \\ y(k) &= G_1(w(k)) \end{aligned} \qquad \begin{aligned} w_x(k+1) &= F_2(w_x(k), P_{w_u(k)}) \\ y(k) &= G_2(w_x(k), P_{w_u(k)}) \end{aligned}$$

$$\begin{aligned} \bar{w}_x(k+1) &= F_3(\bar{w}_x(k), \bar{w}_u(k)) \\ y(k) &= G_3(\bar{w}_x(k), \bar{w}_u(k)) \end{aligned} \qquad \begin{aligned} x(k+1) &= F_4(x(k), u(k)) \\ y(k) &= G_4(x(k), u(k)) \end{aligned}$$

where $F_1 : \mathbf{B}^{2^{n+m}} \to \mathbf{B}^n$, $G_1 : \mathbf{B}^{2^{n+m}} \to \mathbf{B}^p$, $F_2 : \mathbf{B}^{2^n} \times \mathbf{B}^{2^m} \to \mathbf{B}^{2^n}$, $G_2 : \mathbf{B}^{2^n} \times \mathbf{B}^{2^m} \to \mathbf{B}^p$, $F_3 : \mathbf{B}^{2^n} \times \mathbf{B}^{2^m} \to \mathbf{B}^{2^n}$, $G_3 : \mathbf{B}^{2^n} \times \mathbf{B}^{2^m} \to \mathbf{B}^p$, and $F_4 : \mathbf{B}^n \times \mathbf{B}^m \to \mathbf{B}^n$, $G_4 : \mathbf{B}^n \times \mathbf{B}^m \to \mathbf{B}^p$ mappings $k = 0, 1, \ldots$, starting with an initial conditions $w(0)$, $w_x(0)$, $x(0)$. In the Forms 2-3-4, $x \in \mathbf{B}^n$ and $u \in \mathbf{B}^m$ $w_x \in \mathbf{B}^{2^n}$ and $w_u \in \mathbf{B}^{2^m}$ are defined interms of the explicit variables, however, the negation operation is heavily involved. Regardless of this complexity which we will address in the upcoming section, the Forms 2-3-4 depict various properties of Boolean networks in terms of the structure matrices. As it can be observed from the model equations of these different forms, the over all dynamics and the structural properties of Boolean networks are related through system matrices $R$ and $S$ and structure matrices $L$ and $Q$ combined in $(F_i, G_i)$ system pairs, $i = 1, 2, 3, 4$, mappings corresponding to the Forms 1-2-3-4, respectively. In the next subsection, we will provide some properties of these structure matrices to simplify and analyse the properties of above forms of over all Boolean network equations, such as stability, steady state, controllability, observability, state feedback design.

## 3.2 Properties of Structure Matrices

In this section, we will provide some relations involving structure matrices, $L$ and $Q$ ($M$ and $P$ respectively, since $M = \bar{L}^T$, and $Q^T = P$, where we left out the subscript for $x$ and $u$ for the sake of simplicity of our argument). Proofs of the statements will be given selectively and in the form of directions due to space concerns, however they should be quite clear from the tools provided in Section 2 in the form of the lemmata.

**Lemma 3.1.** *Let $L_x$, $M_x$, $L_u$, $M_u$, $P_x$, $Q_x$, $P_u$, and $Q_u$ be structural matrices as defined in the previous section. Then, they possess the following properties:*

$$\begin{bmatrix} M_x & \bar{M}_x \end{bmatrix} \begin{bmatrix} L_x \\ \bar{L}_x \end{bmatrix} = M_x L_x + \bar{M}_x \bar{L}_x = \bar{I}$$

$$\begin{bmatrix} M_u & \bar{M}_u \end{bmatrix} \begin{bmatrix} L_u \\ \bar{L}_u \end{bmatrix} = M_u L_u + \bar{M}_u \bar{L}_u = \bar{I}$$

$$\begin{bmatrix} P_x & P_u \end{bmatrix} \begin{bmatrix} \bar{Q}_x \\ \bar{Q}_u \end{bmatrix} = P_x \bar{Q}_x + P_u \bar{Q}_u = \bar{I}$$