from eigenvalues point of view, and it is left as further research. From many simulations and experimental understanding, since (10)-(11) are in algebraic form, we conjecture that the eigenvalues, λ, of R_x matrix look like satisfying the equationin the form of

$$\lambda^{s_0}(\lambda-1)^{s_1}\prod_{i\in I}(\lambda^{s_{c_i}}-1)^{s_{m_i}}=0,$$

where s_0 represents the number of eigenvalues at 0, $s_1+\sum_{i\in I}s_{m_i}$ is the number of real eigenvalues at 1 on the real axis, and $s_{c_i}-1$ and s_{m_i} represent $s_{c_i}-1$ number of eigenvalues on the unit circle in the complex plane repeated s_{m_i} number of times for each i in some set I forming the related subset of nodes in the Boolean network. It is expected that

$$s_0+s_1+\sum_{i\in I}s_{c_i}s_{m_i}=2^n$$

condition holds.

One can conjecture for trivial and special cases, when R_x is a permutation matrix, or a matrix containing blocks of permutation matrices of different dimensions, that the above condition will hold. This conjecture is yet to be proven for general cases of R_x matrices, and it is left as a further research topic.

4.2.4 Feedback Properties and Feedback Design

Given that there are u and w_u inputs to introduce state and output feedback in Boolean networks, we consider v as the new input and identify that the following vehicles are available to introduce feedback into the three forms, Forms 2-3 and 4, as

$$w_u=K_wP_{w_v}w_x,$$

and

$$u=K_xx+v$$

where K_x is $m\times n$ and K_w is $2^m\times 2^{n+q}$ stochastic matrix, and $v\in\mathbf{B}^q$ is the q dimensional new input. Using Lemma 2.3 and Lemma 2.4, $u=K_xx+v$ feedback law can be restated as

$$\begin{aligned}w_u=&\overline{M_u(K_xL_xQ_x+L_vQ_v)+\bar{M}_u\overline{K_xL_xQ_x+L_vQ_v}}\,w\\=&\overline{M_u(K_xL_xQ_x+L_vQ_v)+\bar{M}_u\overline{K_xL_xQ_x+L_vQ_v}}P_{w_v}w_x.\end{aligned}$$

Therefore, the above two feedback laws are related as

$$K_w=\overline{M_u(K_xL_xQ_x+L_vQ_v)+\bar{M}_u\overline{K_xL_xQ_x+L_vQ_v}}.$$

From practical point of view, the design of feedback laws are naturally done by assigning the input u auto-control through state x feedback, and new controls are added into the network. The above equivalence relations in terms of feedback laws provide an easy and direct way to analyse the effect of feedback at w_u level. It is quite un-natural and impractical way to apply a feedback as w_u is a vector with components which are in conjunctive form of the input u.

To see the effect of feedback laws, if we apply the above feedback laws to Form 2 and Form-4 given in (3)-(4), (14)-(15), respectively, from Form-2, using Lemma 3.2, we obtain:

$$\begin{aligned}w_x(k+1)&=\overline{\bar{M}_x\bar{R}+M_xR}\;P_{K_w}P_{w_v}w_x(k)\\y(k)&=S\;P_{K_w}P_{w_v}w_x(k).\end{aligned}$$