where w is now in conjunctive canonical form of x and v, which implies, as indicated above,

$$K_w = \overline{M_u(K_xL_xQ_x + L_vQ_v) + \bar{M}_u\overline{K_xL_xQ_x + L_vQ_v}}. \tag{31}$$

Equations (29) and (30) are yet to be investigated in the light of the equivalence conditions and relations between Form-1 and Form-4 representations of Boolean networks after feedback law is applied. The question to be investigated further is: given the relationship in (31), whether the equivalance relation for the two forms, Form-1 and Form-4, are invariant under the feedback, if one already exists before the feedback law is applied. The investigation of this point on the effect of feedback in Boolean networks is left as further research topic.

Note that (31) results in an equivalent feedback matrix K_w given K_x feedback matrix. However, given K_w, it is very challenging to find K_x. Instead of working with (31), in general, (29) and (30) can be used to solve for K_w given K_x, and for K_x given K_w. It should be clear that the required solutions can be found using Lemma 2.15 and Lemma 2.16 together with Lemma 3.2.

5 Conclusions and Future Work

We reviewed some results, and developed and provided other results regarding Boolean variables, vectors, matrices, and equations to be used in the analysis and design of Boolean networks. We have presented a model for Boolean networks and their four different forms depicting the state variable and the input analytically and explicitly. We identified structure matrices inherent in Boolean networks. We further specialized the Forms into Boolean conjunctive networks. We derived formulae to transform Form-1 equations into lower dimensional equations in terms of structure matrices of Boolean networks. Due to explicit structure matrices in their equations, the expressions indicate that they can also be used to further explore controllability, observability and feedback concepts for more general Boolean networks. Given that there are u and w_u inputs to introduce state and output feedback in Boolean networks such as, considering v as the new input, the equivalance relations between the two types of feedback constitute a further research topic. We have demonstrated that the controllability, observability, and feedback properties of Boolean networks have very similar appearence however they have qualitatively different conditions and effects on Boolean networks than they do in their continuous and discrete time counter parts. Spectral properties of Boolean matrices are to be investigated further in the light of structure matrices to determine the limit cycles in Boolean networks and their period lengths.

6 References

References

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