Solving Large Scale Optimal Control Problems

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***Abstract*— We present a new method of solving large scale optimal control problems. The most important parts of the algorithm are a method of reducing size of the eigenvalue problem, and an algorithm of solving problems in block upper triangular form. Numerical experiments show, that proposed algorithm can solve optimal control problems with ~10^3 variables in few seconds.**

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I. THE PROBLEM

Consider a system of a linear stochastic optimal control problems in the discrete time. The first order necessary conditions for a solution can be represented as the following linear system (see for example [8]):

$$0 = Ay_t + By_{t+1} + CE_t y_{t+1} + V\varepsilon_t \tag{1}$$

where y is a vector of control variables, ε_t is a vector of i.i.d. random variables with zero mean. Each variable with time subscript t belongs to information set in period t. We assume that matrices A, B, C are square and the matrix pair $(A, B + C)$ is regular. Let $D = B + C$.

Definition 1.1 A matrix pair (A, B) is regular if there exist scalars $\alpha, \beta \in \mathcal{C}$ such that $\det(\alpha A - \beta B) \neq 0$.

We are looking for a solution in the form

$$\begin{aligned} u_t &= Pu_{t-1} + Q\varepsilon_t \\ y_t &= Ru_{t-1} + S\varepsilon_t \end{aligned} \tag{2}$$

where $u \in \mathbb{R}^k$ is vector of state variables. Additionally we require the matrix R is stable. Substituting (2) to (1) yields

$$0 = \Phi_u u_{t-1} + \Phi_\varepsilon^1 \varepsilon_t + \Phi_\varepsilon^2 \varepsilon_{t+1} \tag{3}$$

where matrices Φ_u, Φ_ε^1, and Φ_ε^2 are defined by conditions:

$$\begin{aligned} \Phi_u &= AR + DRP \\ \Phi_\varepsilon^1 &= AS + DRQ + V^1 \\ \Phi_\varepsilon^2 &= BS \end{aligned} \tag{4}$$

equation (3) must be fulfilled for all u_{t-1}, ε_t, ε_{t+1}. Therefore $\Phi_u = 0$, $\Phi_\varepsilon^1 = 0$, and $\Phi_\varepsilon^2 = 0$, and matrices P, Q, R, S forming the solution (2) are determined by matrix equations:

$$0 = \Phi_u = AR + DRP \tag{5}$$

and

$$0 = \Phi_\varepsilon^1 = AS + DRQ + V^1 \quad 0 = \Phi_\varepsilon^2 = BS \tag{6}$$

We call the equation (5) the deterministic part and equations under (6) the stochastic part. In this paper we concentrate on solution to the deterministic part.

Definition 1.2 We call matrices M, N respectively a right range matrix and a right null matrix of a matrix A, if M, N have full column rank, and columns of M, N spans the right range space and the right kernel of the matrix A.

Definition 1.3 We call matrices M, N respectively a left range matrix and a left null matrix of a matrix A, if M, N have full row rank, and rows of M, N spans the left range space and the left kernel of the matrix A.

II. THE BASIC ALGORITHM

In this section we briefly present a method of solving the matrix equation $AR + DRP = 0$ for a regular matrix pair (A, D).

Theorem 2.1 The generalized Schur decomposition. For any square matrices $\mathcal{A}, \mathcal{B} \in R^{n\times n}$ there exist orthogonal matrices Q, Z, and real matrices R_A, R_B, such that R_B is upper-triangular, R_A is quasi-upper triangular and

$$\mathcal{A}Z = QR_A \qquad \mathcal{B}Z = QR_B$$

Additionally, eigenvalues of R_A, R_B can be sorted in any order.

Let us assume that a matrix pair (A, D) is regular1. Let us consider the generalized Schur decomposition of the matrix pair (A, D)

$$AZ = QT_A \qquad DZ = QT_D \tag{7}$$

where matrices Q and Z are orthogonal, the matrix T_A is quasi-upper triangular, and the matrix T_D is upper triangular. Such a decomposition always exists. Let λ_i^A, λ_i^D are i-th eigenvalues of T_A and T_D respectively.

Proposition 2.2 If the matrix pair (A, D) is regular and $\lambda_i^D = 0$, then $\lambda_i^A \neq 0$.

[1] This assumption guarantees that the Schur decomposition is numerically stable. In opposite case a matrix pair (A, D) has infinitely many eigenvalues. Small perturbations of (A, D) may drastically change matrices T_A and T_D.